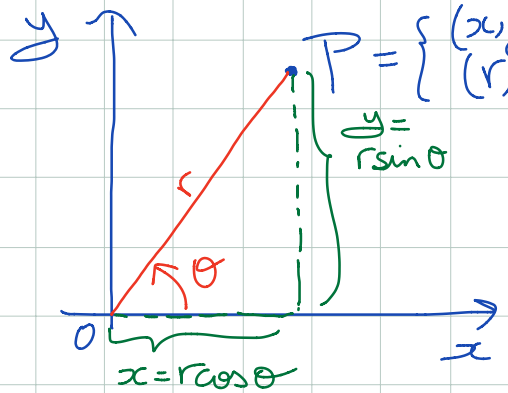
11.3 Polar Coordinates



$P = \begin{cases} (x,y) \rightsquigarrow \text{cartesian/rectangular} \\ (r,\theta) \rightsquigarrow \text{Polar} \end{cases}$

r = distance from the origin (radial coordinate)

θ = angle between $\overline{OP}$ and the positive x-axis. (angular coordinate)

Convention: the angle is positive if the corresponding rotation is counterclockwise

Rectangular to Polar

$r = \sqrt{x^2 + y^2}$ & $\tan\theta = y/x$

Polar to Rectangular

$x = r\cos\theta$ & $y = r\sin\theta$

Examples:

1) Convert $(r,\theta) = (7, \pi/3)$ to Cartesian Coordinates

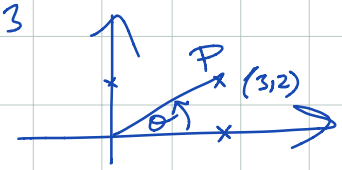
Sol'n: $x = r\cos\theta = 7\cos\pi/3 = 7 \times 1/2 = 7/2$

$y = r\sin\theta = 7\sin\pi/3 = 7 \times \sqrt{3}/2$

$\Rightarrow (x,y) = (7/2, 7\sqrt{3}/2)$

2) Find the polar coordinates of $P = (x,y) = (3,2)$

Sol'n: $r = \sqrt{x^2+y^2} = \sqrt{3^2+2^2} = \sqrt{13}$



$\tan\theta = y/x = 2/3$

$\Rightarrow \theta = \tan^{-1}(y/x) = \tan^{-1}(2/3) \approx 0.588$

(P is in the first quadrant)

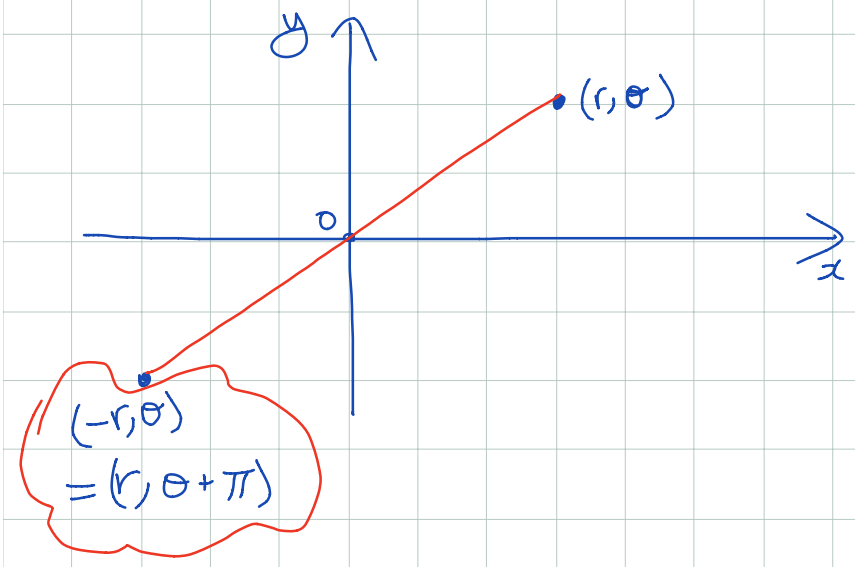
Important Comments

- Angles are not unique

$(r,\theta) = (r, \theta+2\pi) = (r, \theta-2\pi)$

$= (r, \theta+2\pi n)$ for any $n \in \mathbb{Z}$

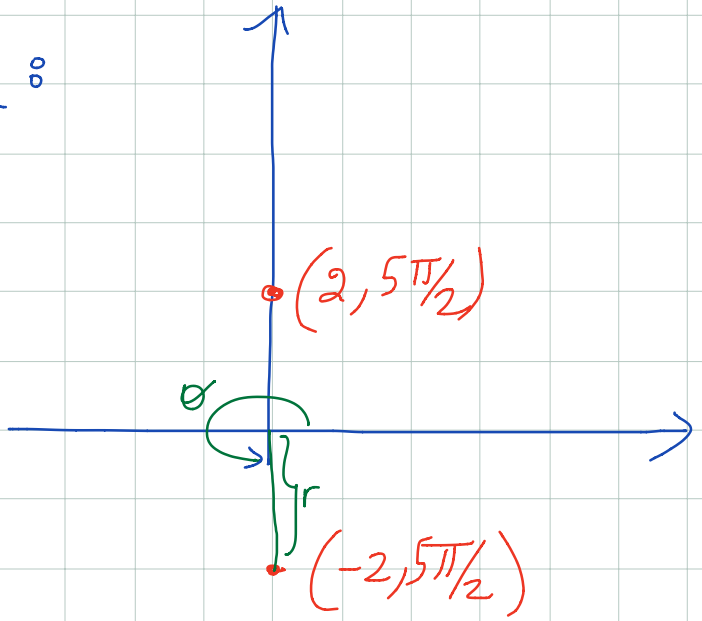
- Origin $= (0, \theta)$ for any angle theta.
- Convention: Allow negative r coordinates



- Aside from the origin, every pt has unique polar coordinate if we insist on $r > 0$ & $0 \leq \theta < 2\pi$

Example: Find a polar representation of $P = (r_0, \theta_0) = (-2, 5\pi/2)$ with $\begin{cases} r > 0 \\ 0 \leq \theta < 2\pi \end{cases}$

Soln:



$$\Longrightarrow \left(-2, \tfrac{5\pi}{2}\right) = \left(2, \tfrac{3\pi}{2}\right)$$

Important: Choosing θ correctly

Recall: If $P = \begin{cases} (x,y) \text{ in cart. coord.} \\ (r,\theta) \text{ in polar coord.} \end{cases}$

then $r = \sqrt{x^2+y^2}$ & $\tan\theta = y/x$ (when $x \neq 0$)

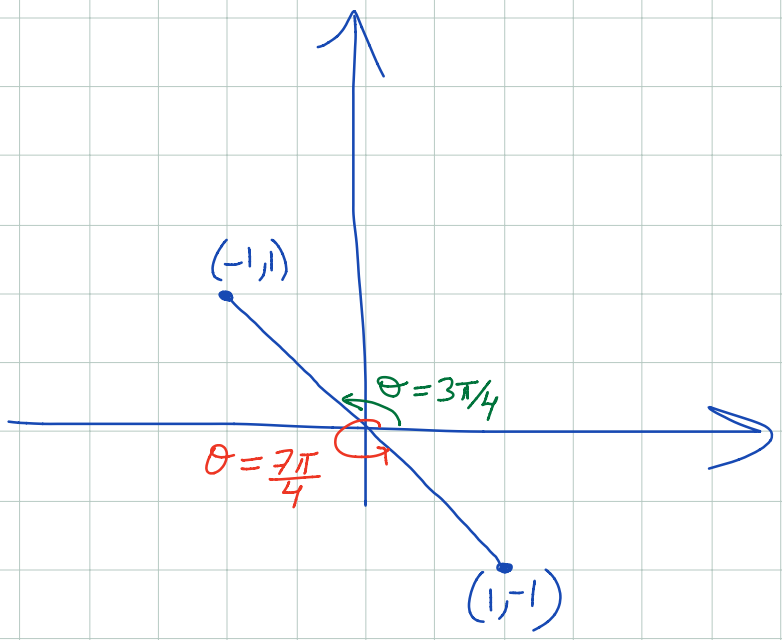
Example: Find the polar coordinates of the two points $(x,y) = (1,-1)$ & $(x,y) = (-1,1)$

Here, note that for both point.

$r = \sqrt{x^2+y^2} = \sqrt{2}$

& $\tan\theta = -1$

but these are different pt.!

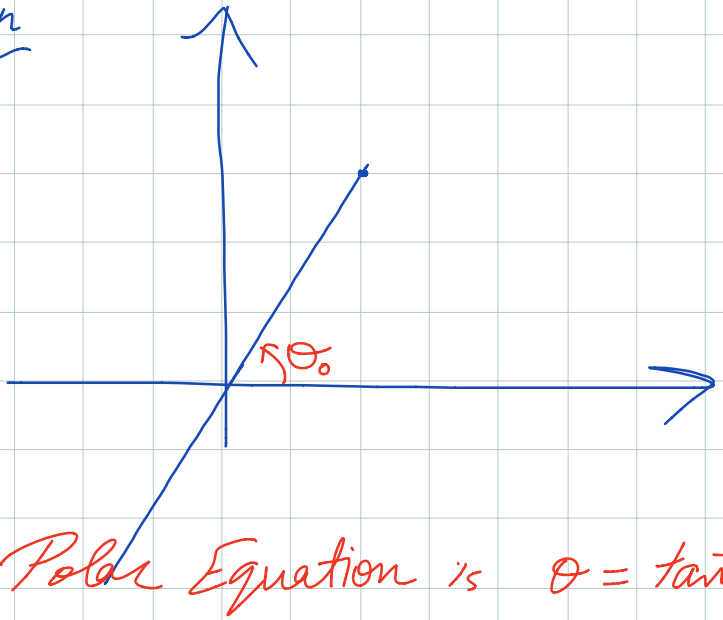


So: Cart.	Polar
(1,-1)	$(\sqrt{2}, 7\pi/4)$
(-1,1)	$(\sqrt{2}, 3\pi/4)$

Equations in Polar Coordinates

Example: Find the polar equation, of the line passing through the origin, with slope $3/2$

Sol'n



Polar Equation is $\theta = \tan^{-1} 3/2$

Example: Identify the curve with polar equation $r = 5$

Sol'n: Circle of radius 5!

Example: Identify the curve with polar equation $r = 2a\cos\theta$ (a: const.)

Soln: we know that $x^2+y^2 = r^2$

& $r = 2a\cos\theta \Rightarrow r^2 = 2a\, r\cos\theta$

($r^2 = x^2+y^2$, $r\cos\theta = x$)

So we have $x^2+y^2 = 2ax$

$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$

$(x-a)^2 + y^2 = a^2$

$\Rightarrow$ circle of radius a & center $(a,0)$.

Similarly: $r = 2a\sin\theta$ is a circle of radius a & center $(0,a)$

Exercise: Write the polar equation of a circle with radius 2 and center $(x, y) = (2, 3)$

Soln: Cartesian equation: $(x-2)^2 + (y-3)^2 = 4$

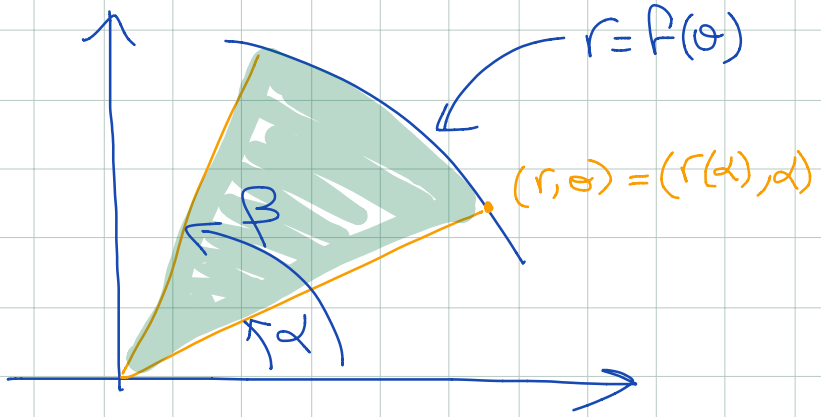
$$\Rightarrow x^2 - 4x + 4 + y^2 - 3y + 9 = 4$$

$$\Rightarrow r^2 - 4r\cos\theta - 3r\sin\theta + 9 = 0$$

$$\Rightarrow r^2 - r(4\cos\theta + 3\sin\theta) + 9 = 0$$

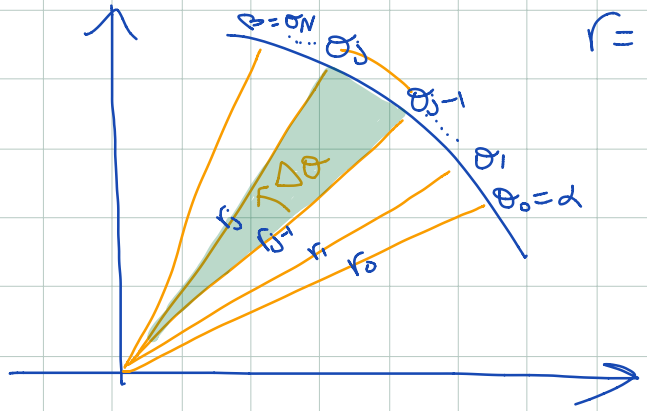
11.4 Area & Arc Length in Polar Coord.

Polar Coord. are convenient for finding the area of a sector bounded by a curve.



How?

Riemann Sums!



$r = f(\theta)$

Area of circular sector of radius r & and angle $\Delta\theta = \frac{r^2}{2}\Delta\theta$

(why? : $A(\text{circle}) = \pi r^2 = \frac{r^2}{2}\underbrace{2\pi}_{\text{full revolution}}$,

you can slice a circle into $\frac{2\pi}{\Delta\theta}$ pieces of angle $\Delta\theta$

$\Rightarrow$ area of each is $\left(\frac{r^2}{2}\cdot 2\pi\right)/\left(\frac{2\pi}{\Delta\theta}\right) = \frac{r^2}{2}\Delta\theta$

$\Rightarrow$ Area of non circular sector (why?)

$$\approx \sum_{i=1}^{N} \frac{1}{2} r_j^2 \Delta\theta$$

$$= \sum_{i=1}^{N} \frac{1}{2} f(\theta_j)^2 \Delta\theta$$

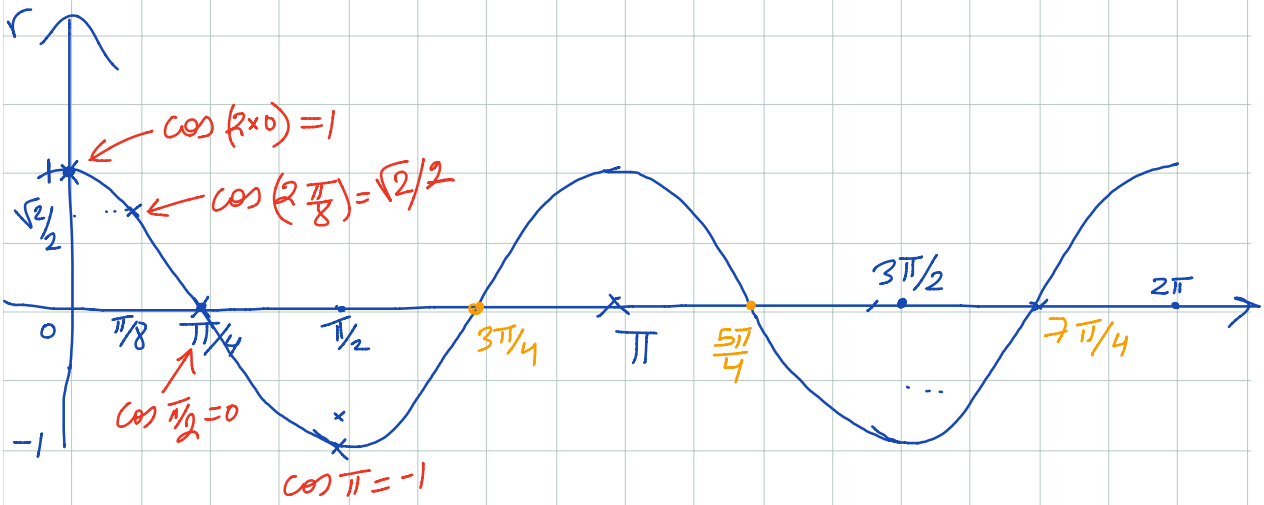
Take limits $\Rightarrow$ $\boxed{A(\text{region}) = \frac{1}{2}\int_\alpha^\beta \underbrace{f(\theta)^2}_{r^2} d\theta}$

Example: The area of a circle of radius 2 is $\frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} 4 d\theta = 2(2\pi - 0)$

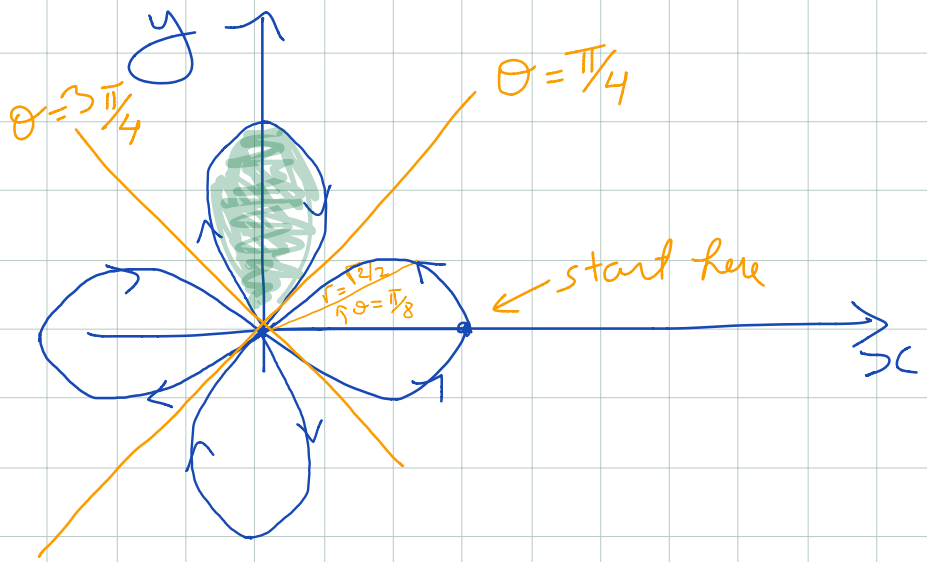
$= 4\pi$

Example: Sketch $r = \cos(2\theta)$ and find the area of one petal

Soln: ① Sketch r Vs θ in cartesian coord.



② "Translate" to polar



③ Compute the area of a petal

$$A = \frac{1}{2}\int_{\pi/4}^{3\pi/4} (\cos 2\theta)^2 \, d\theta$$

Recalling that $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\Rightarrow \cos^2 2\theta = \tfrac{1}{2}(1+\cos 4\theta)$$

$$A = \frac{1}{2}\int_{\pi/4}^{3\pi/4} \frac{1}{2} + \frac{1}{2}\cos 4\theta \, d\theta$$

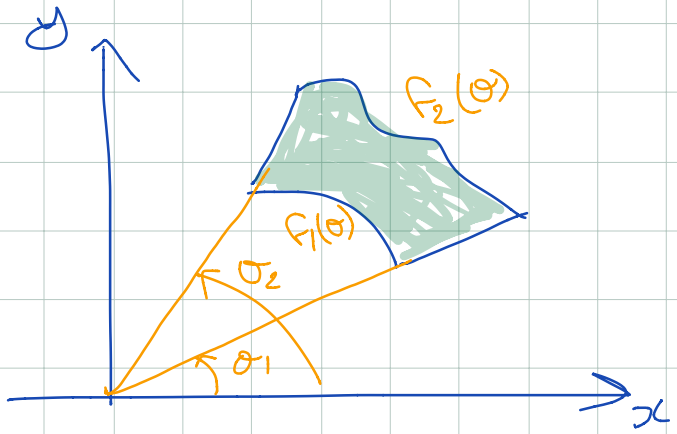
$$= \frac{1}{4}\int_{\pi/4}^{3\pi/4} d\theta + \frac{1}{4}\int_{\pi/4}^{3\pi/4} \cos 4\theta \, d\theta$$

$$= \frac{1}{4}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + \frac{1}{4}\cdot \frac{\sin(4\theta)}{4}\Big|_{\pi/4}^{3\pi/4}$$

$$= \frac{1}{4}\cdot\frac{\pi}{2} + 0 - 0 = \frac{\pi}{8}$$

— x —

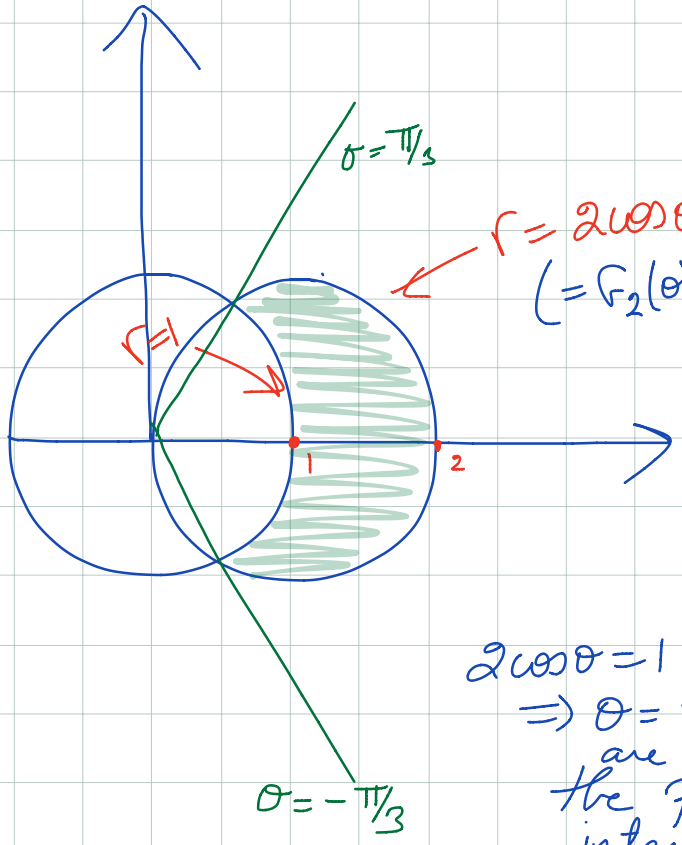
Area between two curves



Area between two curves $= \frac{1}{2}\int_{\theta_1}^{\theta_2} F_2(\theta)^2 - F_1(\theta)^2\, d\theta$

Example: Find the area of the region inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$

Soln: ① Sketch & find the pts of intersection



$2\cos\theta = 1$
$\Rightarrow \theta = \pm \pi/3$ are the pts of intersection

$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} r_2(\theta)^2 - r_1(\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1^2 \, d\theta$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 2\cos 2\theta + 1 \, d\theta = \frac{1}{2}(\sin 2\theta + \theta)\Big|_{-\pi/3}^{\pi/3}$$

$$= \sqrt{3}/2 + \pi/3$$

Complex Numbers (S.1)

Q: How do we solve equations like

$$x^2 + 1 = 0 \quad ?$$

A: Introduce a new number!

"Imaginary number"

i is the number with the property

$$i^2 = -1 \iff i = \sqrt{-1}$$

Complex Number: (e.g., $3+5i$ or $-7+2i$)

is a number of the form $a+bi$ where

a is called the real part & b is called the imaginary part

Complex numbers have many many important applications (Control theory, circuit analysis, signal processing, quantum mechanics (Schrödinger's eq'n)....)

Properties

- **Addition:** (& subtraction) $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$

  example. $(4+3i) - (2-5i) = 2+8i$

- **Multiplication** (like polynomials)

$$(a+bi)(c+di) = ac + adi + bci + bd\underbrace{i^2}_{=-1}$$

$$= ac + (ad+bc)i - bd$$

$$\boxed{(a+bi)(c+di) = (ac-bd) + (ad+bc)i}$$

- **Division:** $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)}$

$$\boxed{\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}}$$

Definition: Complex Conjugate

Given a complex number $\alpha = a+bi$ it's complex conjugate is $\boxed{\bar{\alpha} = a-bi}$

Properties of the Complex Conj

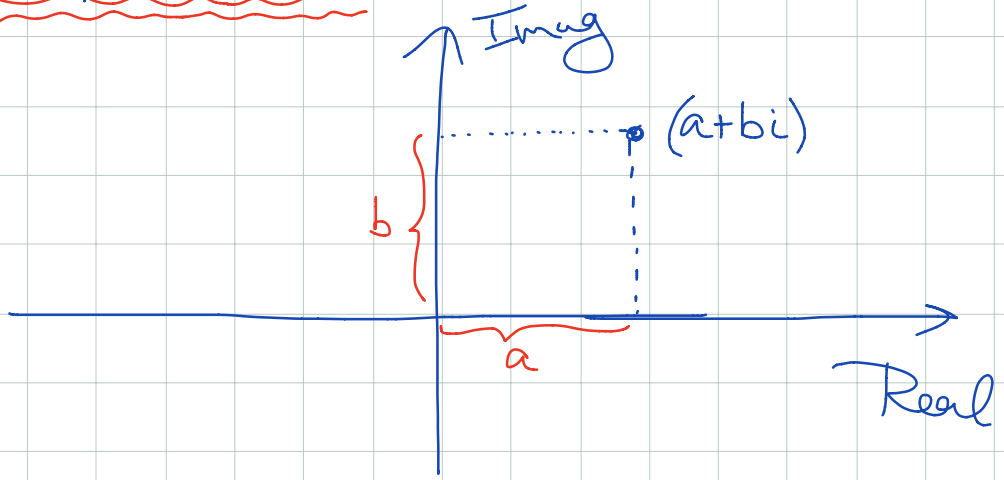
- $\overline{\alpha \pm \beta} = \overline{\alpha} \pm \overline{\beta}$ eg. $\overline{(3+4i)+(4+5i)} = (3-4i) \pm (4-5i)$
- $\overline{\alpha\beta} = \overline{\alpha}\ \overline{\beta}$ eg. $\overline{(1+i)(2+i)} = (1-i)(2-i) = (2+1) - 3i$
- $\overline{\alpha/\beta} = \overline{\alpha}/\overline{\beta}$

Defn: The Magnitude of a complex number

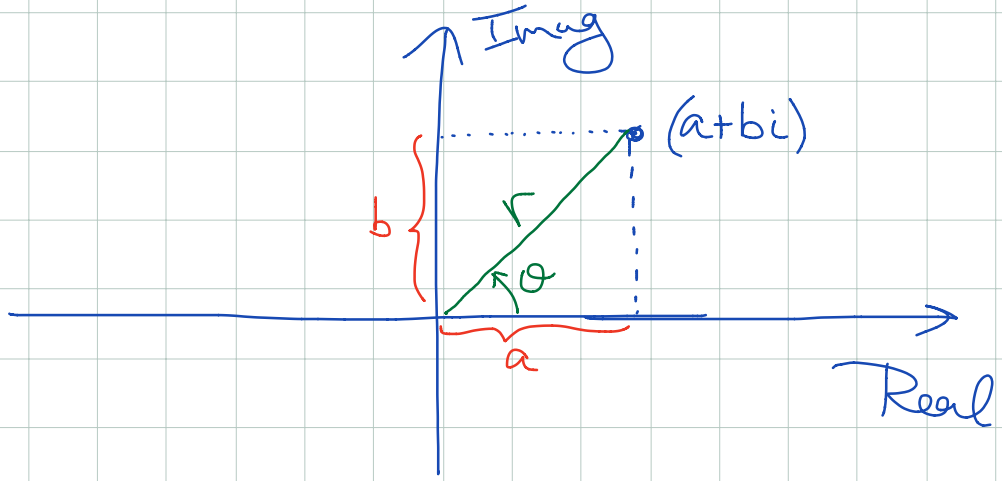
$$|\alpha| = |a+bi| = \sqrt{a^2+b^2}$$

e.g. $|3+4i| = \sqrt{3^2+4^2} = 5$

The Complex Plane



Sometimes convenient to represent complex numbers in "Polar coordinates"



Here $r = \sqrt{a^2+b^2} = |a+bi|$

θ is called the argument of $a+bi$

(written $\theta = \arg(a+bi)$)

Example: Write $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ in Polar Form

Complex Exponentials